On the fundamental bordered matrix of linear estimation

Magnus, J.R.

Published in:
Advanced lectures in quantitative economics

Publication date:
1990

16

On the Fundamental Bordered Matrix of Linear Estimation*

Jan R. Magnus

London School of Economics,
Houghton Street, London WC2A 2AE
and
CentER, Tilburg University,
P.O. Box 90153, 5000 LE Tilburg,
The Netherlands.

1. Introduction

Many problems in economic theory, econometrics, multivariate analysis and mathematical statistics involve the matrix

$$Z = \begin{bmatrix} A & B \\ B' & \mathrm{O} \end{bmatrix}$$

where A is positive semidefinite. The matrix Z has been termed 'the fundamental bordered matrix of linear estimation' [Hall and Meyer (1975)] because of its role in the 'Inverse Partitioned Matrix' method of linear estimation proposed by Rao (1971). Campbell and Meyer (1979) called it 'the fundamental matrix of constrained minimization' because of the importance of Z in the theory of constrained least-squares. The matrix Z also appears in the theory of optimization of definite quadratic forms under linear constraints [Samuelson (1947, pp. 378–9)], comparative statics analysis [Diewert and Woodland (1977)], restricted maximum likelihood estimation

*This paper is a slightly revised version of my 1985 LSE discussion paper [Magnus (1985)]. Some of the results in that paper appeared earlier in Magnus and Neudecker (1988, theorems 3.19–3.24). I am grateful to J. Bout and F. Windmeijer for helpful comments.

ADVANCED LECTURES IN QUANTITATIVE ECONOMICS
ISBN 0-12-711703-2

[Aitchison and Silvey (1958, p. 822), Silvey (1959, pp. 400–2) and Don (1985)], the characterization of estimability [Alalouf and Styan (1979)], and elsewhere.

If A is positive definite (hence non-singular) and B has full column-rank, then Plackett (1960, p. 67) has shown that Z^{-1} exists and takes the form

$$Z^{-1} = \begin{bmatrix} A^{-1} - A^{-1}B(B'A^{-1}B)^{-1}B'A^{-1} & A^{-1}B(B'A^{-1}B)^{-1} \\ (B'A^{-1}B)^{-1}B'A^{-1} & -(B'A^{-1}B)^{-1} \end{bmatrix}.$$

It is well known, however, that Z^{-1} may exist even when A is singular. A necessary and sufficient condition for the non-singularity of Z is that $A + BB'$ is positive definite and B has full column-rank; see Khatri [1968, Lemma 4(i)] or Rao [1973, p. 296]. This case was investigated by Diewert and Woodland (1977).

Very often the matrix Z is known to be singular. For example, in the general linear model

$$y = B\gamma + u, \; \mathrm{E}u = 0, \; \mathrm{E}uu' = \sigma^2 A,$$

the design matrix B may be deficient in rank and the dispersion matrix A may be singular. Rao (1971, 1972) has shown that the problem of inference from the linear model can be completely solved once one has obtained a g-inverse of Z. (For a given $m \times n$ matrix A, any $n \times m$ matrix G satisfying $AGA = A$ is called a generalized inverse (g-inverse) of A and is usually denoted as A^-.) In particular, Rao showed that if the model is consistent and the linear function $W\beta$ is estimable, i.e. if $\mathcal{M}(W') \subset \mathcal{M}(B')$, then

(a) the best linear unbiased estimator of $W\beta$ is $WD_{21}y = WD'_{12}y$
(b) var $(WD_{21}y) = \sigma^2 WD_{22}W'$
(c) an unbiased estimator for σ^2 is given by $\dfrac{y'D_{11}y}{\mathrm{tr}(AD_{11})}$,

where

$$\begin{bmatrix} A & B \\ B' & O \end{bmatrix}^- = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & -D_{22} \end{bmatrix}.$$

Khatri (1968), Pringle and Rayner (1971, pp. 48–52) and Rao (1973, pp. 294–7) obtained properties of Z^- and, in particular, established interconnections between the various blocks of Z^-. Hall and Meyer (1975) showed that the blocks of Z^- are completely independent of each other, while Hall (1975) extended these results to the infinite dimensional setting.

Additional results were obtained by Campbell and Meyer (1979, pp. 63–9 and 104–15).

More general matrices than Z have also been considered. Thus Hall (1976) and Hall and Hartwig (1976) studied the bordered matrix

$$W = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

and obtained necessary and sufficient conditions for the blocks of W^- to be independent, while Mitra (1982) considered the matrix

$$W_o = \begin{bmatrix} A & B \\ C & O \end{bmatrix}$$

with the rank conditions

$$\mathrm{r}(W_o) = \mathrm{r}(A : B) + \mathrm{r}(C) = \mathrm{r}(A' : C') + \mathrm{r}(B),$$

and showed that many of the established results do not depend on the fact that A is a symmetric positive semidefinite matrix. Mitra's results were generalized by Rao and Yanai (1985).

The purpose of this paper is to present properties of Z^+, the Moore–Penrose (MP) inverse of Z. The MP-inverse is *unique* and has therefore certain theoretical and pedagogical advantages over non-unique generalized inverses. Original and complete proofs are given in a unified treatment of the subject. In addition to some well-known results, many new results are presented. An attempt is made to be exhaustive, but without going into detail as regards trivial corollaries.

In Section 2 some properties of the MP-inverse are summarized and one proposition is stated for later application. In Section 3 we present in a series of eight lemmas the most important properties of Z^+. Sections 4 and 5 contain special cases. Finally, in Section 6, we apply the theory to matrix equations and restricted least-squares.

2. Notation and Preliminaries

Matrices are denoted by capital letters, vectors and scalars by lower case letters, and all are assumed to be real. An $m \times n$ matrix is one having m rows and n columns. A' denotes the transpose of A and $\mathrm{r}(A)$ its rank; if A is square $n \times n$, $\mathrm{tr}\, A$ denotes its trace, $|A|$ its determinant, and

$\{\lambda_i(A), i=1,\ldots,n\}$ the set of its eigenvalues. The $n \times n$ identity matrix is denoted I_n.

We recall that the Moore–Penrose inverse of an $m \times n$ matrix A, is the $n \times m$ matrix G satisfying the four conditions

$$AGA = A,\ GAG = G,\ (AG)' = AG,\ (GA)' = GA,$$

and is denoted by A^+; see Moore (1920), Moore (1935, p. 197), and Penrose (1955). The following well-known properties of the MP-inverse are stated here for easy reference:

$$A'(AA')(AA')^+ = A' \tag{2.1}$$

$$AB = 0 \leftrightarrow B^+A^+ = 0 \tag{2.2}$$

and

$$(AB)(AB)^+ = AA^+ \text{ if } |BB'| \neq 0. \tag{2.3}$$

The linear space spanned by the columns of A (the *column-space* of A) is denoted $\mathcal{M}(A)$. We have

$$\mathcal{M}(A) = \mathcal{M}(AA') \tag{2.4}$$

for any matrix A. Finally we note that

$$\mathcal{M}(B) \subset \mathcal{M}(A) \leftrightarrow \mathrm{r}(A:B) = \mathrm{r}(A) \leftrightarrow AA^+B = B. \tag{2.5}$$

Let us now present seven equivalent characterizations of matrices A and B satisfying $\mathrm{r}(A:B) = \mathrm{r}(A) + \mathrm{r}(B)$. Proposition 1, which we shall have opportunity to use later, extends the results obtained by Chipman (1964) for complementary matrices; see also Marsaglia and Styan (1974) and Hall and Hartwig (1976).

Proposition 1. Let A and B be two matrices with the same number of rows. Then the following seven statements are equivalent:

(i) $\mathcal{M}(A) \cap \mathcal{M}(B) = \{0\}$
(ii) $\mathrm{r}(AA' + BB') = \mathrm{r}(A) + \mathrm{r}(B)$
(iii) $A'(AA' + BB')^+A$ is idempotent
(iv) $A'(AA' + BB')^+A = A^+A$
(v) $B'(AA' + BB')^+B$ is idempotent
(vi) $B'(AA' + BB')^+B = B^+B$
(vii) $A'(AA' + BB')^+B = 0$.

Proof. See Magnus and Neudecker (1988, theorem 3.19).

3. Eight lemmas

We shall consider the bordered matrix

$$Z = \begin{bmatrix} A & B \\ B' & 0 \end{bmatrix} \tag{3.1}$$

where A is a real positive semidefinite $n \times n$ matrix and B is a real $n \times k$ matrix. Its Moore–Penrose inverse Z^+ is partitioned conformably as

$$Z^+ = \begin{bmatrix} D & E \\ E' & -F \end{bmatrix}. \tag{3.2}$$

In a sequence of eight lemmas, we shall obtain expressions for D, E and F and study their properties. It is useful to introduce the following four auxiliary matrices:

$$\begin{aligned} N &= A + BB', & M &= I - BB^+, \\ C &= B'N^+B, & R &= MAM. \end{aligned} \tag{3.3}$$

Lemma 1. The matrices N and C satisfy

(i) $\mathcal{M}(A) \subset \mathcal{M}(N)$, $\mathcal{M}(B) \subset \mathcal{M}(N)$, $\mathcal{M}(B') = \mathcal{M}(C)$
(ii) $NN^+A = A$, $NN^+B = B$
(iii) $C^+C = B^+B$, $\mathrm{r}(C) = \mathrm{r}(B)$.

Note. Lemma 1 is also given in Magnus and Neudecker (1988, theorem 3.20).

Proof. Letting $A = TT'$ and recalling from (2.4) that $\mathcal{M}(Q) = \mathcal{M}(QQ')$ for any matrix Q, we have

$$\mathcal{M}(A) = \mathcal{M}(T) \subset \mathcal{M}(T{:}B) = \mathcal{M}(TT' + BB') = \mathcal{M}(N),$$

and similarly $\mathcal{M}(B) \subset \mathcal{M}(N)$. Thus $NN^+A = A$ and $NN^+B = B$. Next, let $N^+ = FF'$ and define $G = B'F$. Then $C = GG'$. Using (ii) and the fact that $G'(GG')(GG')^+ = G'$ for any G [see (2.1)], we get

$$B(I - CC^+) = NN^+B(I - CC^+) = NFG'(I - GG'(GG')^+) = 0,$$

and hence $\mathcal{M}(B') \subset \mathcal{M}(C)$. Since obviously $\mathcal{M}(C) \subset \mathcal{M}(B')$, we find $\mathcal{M}(B') = \mathcal{M}(C)$. Finally, to prove (iii), we note that $\mathcal{M}(B') = \mathcal{M}(C)$ implies that $\mathrm{r}(B') = \mathrm{r}(C)$ and hence that $\mathrm{r}(B) = \mathrm{r}(C)$; it also implies that C can be written as $C = B'Q$ for some non-singular matrix Q. Hence, by (2.3)

$$C^+C = CC^+ = B'Q(B'Q)^+ = B'(B')^+ = B^+B.$$

This concludes the proof. ||

Lemma 2. The matrices M and R satisfy

(i) $RB = R^+B = 0$
(ii) $R^+M = R^+$
(iii) $MAR^+ = RR^+$
(iv) $MAR^+A = MA$.

Proof. Since $MB = 0$ it follows that $RB = 0$, and hence using (2.2) that

$$B'R^+ = B'BB^+R^+ = 0.$$

This proves (i). (ii) is a direct consequence of (i). To prove (iii) we write

$$MAR^+ = MAMR^+ = RR^+,$$

using (ii) and the symmetry of R and M. Finally, let $A = TT'$ and $P = MT$. Then

$$\begin{aligned} MAR^+A &= RR^+A = RR^+MA = PP'(PP')^+PT' \\ &= PT' = MTT' = MA. \; \| \end{aligned}$$

Lemma 3. The Moore–Penrose inverse of Z is

$$Z^+ = \begin{bmatrix} D & E \\ E' & -F \end{bmatrix}$$

where

$$\begin{aligned} D &= N^+ - N^+BC^+B'N^+ = R^+ \\ E &= N^+BC^+ = (I - R^+A)(B^+)' \\ F &= C^+ - CC^+ = B^+(A - AR^+A)(B^+)'. \end{aligned}$$

Note. The first set of expressions for D, E and F is implicit in Rao (1973, p. 296) and explicit in Pringle and Rayner (1971, theorem 3.5) for the special case where A is positive semidefinite; the second set is given by Hall (1975, theorem 4.3) and Don (1985). See also Magnus and Neudecker (1988, theorem 3.21).

Proof. We define

$$Z^* = \begin{bmatrix} N^+ - N^+BC^+B'N^+ & N^+BC^+ \\ C^+B'N^+ & -C^+ + CC^+ \end{bmatrix},$$

so that

$$ZZ^* = \begin{bmatrix} AN^+ - AN^+BC^+B'N^+ + BC^+B'N^+ & AN^+BC^+ - BC^+ + BCC^+ \\ B'N^+ - B'N^+BC^+B'N^+ & B'N^+BC^+ \end{bmatrix}$$

$$= \begin{bmatrix} NN^+ & 0 \\ 0 & CC^+ \end{bmatrix},$$

using $A = N - BB'$, the definition of C, and the results $NN^+B = B$ and $CC^+B' = B'$; see Lemma 1. Since Z and Z^* are both symmetric, it follows that $ZZ^* = Z^*Z$ and that this matrix is also symmetric. It is then clear that $ZZ^*Z = Z$ and $Z^*ZZ^* = Z^*$ and hence that Z^* is the Moore–Penrose inverse of Z.

Next, we define

$$Z^{**} = \begin{bmatrix} R^+ & (I - R^+A)(B^+)' \\ B^+(I - AR^+) & -B^+(A - AR^+A)(B^+)' \end{bmatrix},$$

and obtain

$$ZZ^{**} = \begin{bmatrix} BB^+ + MAR^+ & M(A - AR^+A)(B^+)' \\ B'R^+ & B^+B - B'R^+A(B^+)' \end{bmatrix}$$

$$= \begin{bmatrix} BB^+ + RR^+ & 0 \\ 0 & B^+B \end{bmatrix},$$

using the results of Lemma 2. Again, since Z^{**} is symmetric, it follows that ZZ^{**} and $Z^{**}Z$ are symmetric and equal. An easy application of the results of Lemma 2 then yields $ZZ^{**}Z = Z$ and $Z^{**}ZZ^{**} = Z^{**}$. Hence

$$Z^+ = Z^* = Z^{**}. \quad \|$$

Lemma 4. We have

(i) $ZZ^+ = Z^+Z = \begin{bmatrix} NN^+ & 0 \\ 0 & CC^+ \end{bmatrix} = \begin{bmatrix} BB^+ + RR^+ & 0 \\ 0 & B^+B \end{bmatrix}$

(ii) the rank of Z is

$$\mathrm{r}(Z) = \mathrm{r}(A:B) + \mathrm{r}(B)$$

(iii) Z is non-singular if and only if B has full column-rank and $A + BB'$ is positive definite

(iv) the determinant of Z is

$$|Z| = (-1)^k |C| \; |N| .$$

Note. The fact that ZZ^+ and Z^+Z are block-diagonal was also noted by Hall (1975).

Proof. (i) follows from Lemma 3, (ii) follows from (i), and (iii) follows from (ii). To prove (iv) we notice that

$$\begin{bmatrix} I_n & \frac{1}{2}B \\ 0 & I_k \end{bmatrix} \begin{bmatrix} A & B \\ B' & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ \frac{1}{2}B' & I_k \end{bmatrix} = \begin{bmatrix} N & B \\ B' & 0 \end{bmatrix}$$

and also

$$\begin{bmatrix} I_n & 0 \\ B'N^+ & -I_k \end{bmatrix} \begin{bmatrix} N & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} N & B \\ B' & 0 \end{bmatrix} .$$

Hence,

$$\begin{vmatrix} A & B \\ B' & 0 \end{vmatrix} = \begin{vmatrix} N & B \\ B' & 0 \end{vmatrix} = (-1)^k |N| \; |C| . \;\|$$

Lemma 5. (Properties of C). The positive semidefinite $k \times k$ matrix C satisfies

(i) $0 \leqslant \lambda_i(C) \leqslant 1$, $i = 1, \ldots, k$

(ii) the zero eigenvalue has multiplicity $k - \mathrm{r}(B)$

(iii) the unit eigenvalue has multiplicity $\mathrm{r}(A:B) - \mathrm{r}(A)$

(iv) C is idempotent if and only if $\mathrm{r}(A:B) = \mathrm{r}(A) + \mathrm{r}(B)$, in which case $C = B^+B$

(v) $C = I_k$ if and only if $\mathrm{r}(A:B) = \mathrm{r}(A) + k$.

Proof. The eigenvalues of $C = B'N^+B$ are the same as those of N^+BB', apart from n-k zeroes which belong to the latter matrix. Let λ be an eigenvalue of N^+BB' and let x be the associated eigenvector, so that

$$N^+BB'x = \lambda x. \tag{3.4}$$

Then, since $NN^+B = B$ (Lemma 1), we obtain

$$BB'x = NN^+BB'x = \lambda Nx = \lambda(A + BB')x.$$

If $B'x = 0$ then $\lambda = 0$, by (3.4). If, on the other hand, $B'x \neq 0$ then

$$\lambda = \frac{x'BB'x}{x'(A + BB')x} \leqslant 1$$

with equality if and only if $Ax = O$. This proves (i). The proof of (ii) is trivial because $\mathrm{r}(C) = \mathrm{r}(B)$ by Lemma 1. To prove (iii) we notice first that $\lambda = 1$ if and only if $Ax = 0$, $B'x \neq 0$. The number of linearly independent x satisfying $Ax = 0$ is $n - \mathrm{r}(A)$. The number of linearly independent x satisfying $Ax = 0$ *and* $B'x = 0$ (i.e. $Nx = 0$) is $n - \mathrm{r}(N)$. Hence the number of linearly independent x satisfying $Ax = 0$, $B'x \neq 0$ is

$$(n - \mathrm{r}(A)) - (n - \mathrm{r}(N)) = \mathrm{r}(N) - \mathrm{r}(A) = \mathrm{r}(A:B) - \mathrm{r}(A).$$

This is therefore the multiplicity of the unit eigenvalue.

Finally, (iv) follows from Proposition 1, and (v) follows from (iii). ||

Lemma 6. The following relationships hold:

(i) $AD + BE' = NN^+$
(ii) $DB = 0$, $D^+B = 0$
(iii) $AE = BF$
(iv) $F = E'AE$
(v) $NN^+D = D$, $NN^+E = E$
(vi) $BE'B = B$, $E'BE' = E'$, $(E'B)' = E'B = B^+B$
(vii) $\mathrm{r}(E) = \mathrm{r}(B)$
(viii) $E' = B^+$ if and only if $MAB = 0$
(ix) $DAD = D$
(x) $BB^+ + DD^+ = NN^+$
(xi) $MD = D$, $MAD = D^+D$, $MADA = MA$
(xii) $C^+ = B^+(N - ADA)(B^+)'$.

Proof. From Lemma 4(i) we have

$$\begin{bmatrix} A & B \\ B' & 0 \end{bmatrix} \begin{bmatrix} D & E \\ E' & -F \end{bmatrix} = \begin{bmatrix} NN^+ & 0 \\ 0 & B^+B \end{bmatrix}$$

and hence $AD + BE' = NN^+$, $B'D = 0$ (so that $DB = D^+B = 0$), and $AE = BF$. This proves (i)–(iii). To prove (iv) we write, using (iii) and Lemma 3,

$$\begin{aligned} E'AE = E'BF &= C^+B'N^+B(C^+ - CC^+) \\ &= C^+C(C^+ - C^+C) = C^+ - C^+C = F. \end{aligned}$$

To prove (v) it suffices to note from Lemma 3 that $D = N^+P$ and $E = N^+Q$ for some P and Q and that $N = N'$ (so that $NN^+ = N^+N$). Next, since $E = N^+BC^+$ (Lemma 3), we obtain

$$E'B = C^+B'N^+B = C^+C = B^+B,$$

using Lemma 1(iii). Hence $BE'B = B$ and

$$E'BE' = B^+BC^+B'N^+ = C^+B'N^+ = E'.$$

(vii) follows from (vi). Next, to prove (viii) we use $E = (I - R^+A)(B^+)'$ from Lemma 3, so that $E = B^{+\prime} \leftrightarrow R^+AB^{+\prime} = 0 \leftrightarrow RR^+AB = 0 \leftrightarrow (NN^+ - BB^+)AB = 0 \leftrightarrow (I - BB^+)AB = 0 \leftrightarrow MAB = 0$, using the facts $NN^+ = BB^+ + RR^+$ (Lemma 4(i)) and $NN^+A = A$ (Lemma 1(ii)). (ix) follows from $D = N^+ - N^+BC^+B'N^+$ (Lemma 3), using $N^+AN^+ = N^+ - N^+BB'N^+$ and $BCC^+ = B$. (x) follows from Lemma 4(i) and $D = R^+$ (Lemma 3). To demonstrate (xi) we use $DB = 0$ to prove $MD = D$, and Lemma 2 (iii) + (iv) and $D = R^+$ to prove $MAD = D^+D$ and $MADA = MA$. Finally, to prove (xii) we use (from Lemma 3)

$$C^+ - CC^+ = B^+(A - ADA)(B^+)',$$

so that

$$\begin{aligned} C^+ &= B^+(A - ADA)(B^+)' + CC^+ \\ &= B^+(A - ADA)(B^+)' + B^+B \\ &= B^+(A - ADA + BB')(B^+)' \\ &= B^+(N - ADA)(B^+)'. \;\| \end{aligned}$$

Lemma 7. (Properties of F). The $k \times k$ matrix F satisfies

(i) F is symmetric and positive semidefinite
(ii) $\mathrm{r}(F) = \mathrm{r}(A) + \mathrm{r}(B) - \mathrm{r}(A:B)$
(iii) $\mathrm{r}(F) \leqslant \mathrm{r}(B)$ with equality if and only if $\mathcal{M}(B) \subset \mathcal{M}(A)$
(iv) F is positive definite if and only if $\mathcal{M}(B) \subset \mathcal{M}(A)$ and $\mathrm{r}(B) = k$

(v) $F = 0$ if and only if $\mathrm{r}(A:B) = \mathrm{r}(A) + \mathrm{r}(B)$
(vi) $F^+ = C(I - C)^+$
(vii) the positive eigenvalues of F are $-1 + 1/\lambda_i$, where λ_i $(i = 1, 2, \ldots, \mathrm{r}(F))$ are the eigenvalues of C satisfying $0 < \lambda_i < 1$.

Proof. From the representation $F = C^+ - CC^+$ (see Lemma 3) it is clear that F is symmetric. Let S be an orthogonal matrix such that

$$S'CS = \Lambda \tag{3.5}$$

where $\Lambda = \text{diag}\ (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a diagonal matrix containing the eigenvalues of C. Then

$$S'FS = S'(C^+ - CC^+)S = \Lambda^+ - \Lambda\Lambda^+.$$

The eigenvalues $\mu_1, \ldots, \mu_k$ of F are thus

$$\mu_i = \lambda_i^+ - \lambda_i\lambda_i^+ \qquad (i = 1, \ldots, k), \tag{3.6}$$

and since $0 \leqslant \lambda_i \leqslant 1$ (Lemma 5(i)), we obtain

$$\mu_i \geqslant 0 \qquad (i = 1, \ldots k).$$

Hence F is positive semidefinite. To prove (ii) we note that the rank of F is equal to the number of positive eigenvalues its possesses. If $\lambda_i = 0$ or $\lambda_i = 1$ then $\mu_i = 0$; if $0 < \lambda_i < 1$ then $\mu_i > 0$. Hence, by Lemma 5(ii) and (iii),

$$\begin{aligned}\mathrm{r}(F) &= k - (k - \mathrm{r}(B)) - (\mathrm{r}(A:B) - \mathrm{r}(A))\\ &= \mathrm{r}(A) + \mathrm{r}(B) - \mathrm{r}(A:B).\end{aligned}$$

This proves (ii). Properties (iii), (iv) and (v) are direct consequences of (ii). To prove (vi) we use the diagonal representation (3.5) and write $S'CS = \Lambda$ so that $S'FS = (\mathrm{I} - \Lambda)\Lambda^+$. Then $S'F^+S = ((I - \Lambda)\Lambda^+)^+ = \Lambda(I - \Lambda)^+$ and the result follows. Finally, the $\mathrm{r}(F)$ positive eigenvalues of F are given by (3.6) for all $0 < \lambda_i < 1$. Hence (vii) follows. ||

Lemma 8. (Properties of D). The $n \times n$ matrix D satisfies

(i) D is symmetric and positive semidefinite
(ii) $\mathrm{r}(D) = \mathrm{r}(A:B) - \mathrm{r}(B)$
(iii) $\mathrm{r}(D) \leqslant \mathrm{r}(A)$ with equality if and only if $\mathrm{r}(A:B) = \mathrm{r}(A) + \mathrm{r}(B)$
(iv) D is positive definite if and only if A is positive definite and $\mathrm{r}(A:B) = \mathrm{r}(A) + \mathrm{r}(B)$

(v) $D = 0$ if and only if $\mathscr{M}(A) \subset \mathscr{M}(B)$
(vi) $D^+ = MAM$, where $M = I - BB^+$
(vii) $D^+ = A$ if and only if $AB = 0$
(viii) $D = \sum_{i=1}^{r(D)} \lambda_i^{-1} x_i x_i'$, where $\lambda_1, \lambda_2, \ldots, \lambda_{r(D)}$ are the positive eigenvalues of MAM and $x_1, x_2, \ldots, x_{r(D)}$ are the corresponding normalized eigenvectors
(ix) if $r(B) = k$, then the determinantal equation

$$\phi(\lambda) = \begin{vmatrix} A - \lambda I & B \\ B' & 0 \end{vmatrix} = 0$$

has $n - k$ roots of which $n - r(A:B)$ are zero and $r(A:B) - k$ are positive; the reciprocals of these positive roots are equal to the positive eigenvalues of D.

Note. Property (ix) was proved by Diewert and Woodland (1977, p. 393) for the case $r(A:B) = n$.

Proof. D is symmetric because Z and therefore Z^+ are symmetric; its positive semidefiniteness follows from $DAD = D$ (Lemma 6(ix)). To determine the rank of D we recall from Lemma 6(x) that $DD^+ = NN^+ - BB^+$. Hence

$$\begin{aligned} r(D) &= r(DD^+) = \operatorname{tr}(DD^+) = \operatorname{tr} NN^+ - \operatorname{tr} BB^+ \\ &= r(N) - r(B) = r(A:B) - r(B). \end{aligned}$$

Properties (iii)–(v) are direct consequences of (ii). Since $D = R^+$ (Lemma 3), we obtain $D^+ = R = MAM$. This proves (vi). (vii) is a special case of (vi): $D^+ = A \Rightarrow MAM = A \Rightarrow AB = MAMB = 0$, and conversely $AB = 0 \Rightarrow AM = MA = A \Rightarrow MAM = A$. (viii) follows from (vi). Finally, to prove (ix) it suffices to show that λ is a positive eigenvalue of MAM if and only if λ is a nonzero root of $\phi(\lambda) = 0$. Assume first that $MAMx = \lambda x$ for some $\lambda > 0$ and $x \neq 0$. Then $Mx = x$ and $B'x = 0$, so that

$$\begin{bmatrix} A - \lambda I & B \\ B' & 0 \end{bmatrix} \begin{bmatrix} I \\ -B^+A \end{bmatrix} x = \begin{bmatrix} MAx - \lambda x \\ B'x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Hence $\phi(\lambda) = 0$. Assume next that $\phi(\lambda) = 0$ for some $\lambda \neq 0$. Then there exist vectors x and y, not both zero, such that

$$\begin{bmatrix} A - \lambda I & B \\ B' & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

or equivalently

$$Ax + By = \lambda x \tag{3.7}$$

$$B'x \qquad = 0. \tag{3.8}$$

If $x = 0$, then (3.7) implies $By = 0$ and hence $y = 0$, since $\mathrm{r}(B) = k$. Hence $x \neq 0$. Premultiplying (3.7) by M yields $MAx = \lambda Mx$, and since $Mx = x$ by (3.8), we obtain $MAMx = \lambda x$. This concludes the proof. ||

4. Three Special Cases

There are three special cases which are of particular importance. Since

$$\mathrm{r}(A) \leqslant \mathrm{r}(A:B) \leqslant \mathrm{r}(A) + \mathrm{r}(B),$$

it makes sense to study the two polar cases

$$\mathrm{r}(A:B) = \mathrm{r}(A) \tag{4.1}$$

and

$$\mathrm{r}(A:B) = \mathrm{r}(A) + \mathrm{r}(B). \tag{4.2}$$

The first of these extreme cases arises when the columns of B are linear combinations of the columns of A, that is $\mathscr{M}(B) \subset \mathscr{M}(A)$, while the second case arises when the column spaces of A and B have only the origin in common. A further special case is obtained by replacing (4.2) by the stronger condition $AB = 0$, which amounts to orthogonality of the column spaces of A and B.

The third case of importance is

$$(I - BB^+)AB = 0, \tag{4.3}$$

and it produces two interesting corollaries: $AB = 0$ and $\mathscr{M}(A) \subset \mathscr{M}(B)$. Thus, the condition $AB = 0$ arises as a special case of both (4.2) and (4.3).

Lemmas 9–11 deal with the special cases (4.1)–(4.3) respectively.

Lemma 9. In the special case when $\mathscr{M}(B) \subset \mathscr{M}(A)$, we have

$$Z^+ = \begin{bmatrix} A^+ - A^+B\Gamma^+B'A^+ & A^+B\Gamma^+ \\ \Gamma^+B'A^+ & -\Gamma^+ \end{bmatrix},$$

where $\Gamma = B'A^+B$, and

$$ZZ^+ = Z^+Z = \begin{bmatrix} AA^+ & 0 \\ 0 & B^+B \end{bmatrix}.$$

Further,

(i) $AA^+B = B$, $\Gamma\Gamma^+ = B^+B$
(ii) Γ and Γ^+ are symmetric and positive semidefinite with $r(\Gamma) = r(\Gamma^+) = r(B)$
(iii) $A^+ - A^+B\Gamma^+B'A^+$ is symmetric and positive semidefinite with rank $r(A) - r(B)$
(iv) $r(Z) = r(A) + r(B)$
(v) $|Z| \neq 0$ if and only if A is positive definite and $r(B) = k$
(vi) $|Z| = (-1)^k |A|\ |\Gamma|$
(vii) $R^+ = A^+ - A^+B\Gamma^+B'A^+$, $\quad RR^+ = AA^+ - BB^+$, $\quad A - AR^+A = B\Gamma^+B'$, where R denotes the $n \times n$ matrix $(I - BB^+)A(I - BB^+)$.

Note. The expressions for Z^+ and ZZ^+ and property (i) were also presented in Magnus and Neudecker (1988, theorem 3.22).

Proof. The first statement of (i) follows from $\mathcal{M}(B) \subset \mathcal{M}(A)$. To prove the second statement of (i) we write $A = TT'$ with $|T'T| \neq 0$ and $B = TS$. Then

$$\Gamma = B'A^+B = S'T'(TT')^+TS = S'S \tag{4.4}$$

so that, using (2.3),

$$B^+B = (TS)^+(TS) = S^+S = (S'S)^+S'S = \Gamma^+\Gamma = \Gamma\Gamma^+.$$

As a consequence we also have $\Gamma\Gamma^+B' = B'$. Now, let Z^* be defined by

$$Z^* = \begin{bmatrix} A^+ - A^+B\Gamma^+B'A^+ & A^+B\Gamma^+ \\ \Gamma^+B'A^+ & -\Gamma^+ \end{bmatrix}.$$

Then,

$$ZZ^* = \begin{bmatrix} AA^+ - AA^+B\Gamma^+B'A^+ + B\Gamma^+B'A^+ & AA^+B\Gamma^+ - B\Gamma^+ \\ B'A^+ - B'A^+B\Gamma^+B'A^+ & B'A^+B\Gamma^+ \end{bmatrix}$$
$$= \begin{bmatrix} AA^+ & 0 \\ 0 & \Gamma\Gamma^+ \end{bmatrix} = \begin{bmatrix} AA^+ & 0 \\ 0 & B^+B \end{bmatrix},$$

using the facts $AA^+B = B$, $\Gamma\Gamma^+B' = B'$, and $\Gamma\Gamma^+ = B^+B$. To show that $Z^* = Z^+$ is then straightforward.

To prove (ii) it suffices to note, from (4.1), that $\mathrm{r}(\Gamma) = \mathrm{r}(S) = \mathrm{r}(B)$; (iii) follows from Lemma 8 (i) and (ii); (iv) follows from Lemma 4(ii); (v) follows from (iv); (vi) follows from the factorization

$$\begin{bmatrix} A & B \\ B' & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ B'A^+ & -I_k \end{bmatrix} \begin{bmatrix} A & B \\ 0 & \Gamma \end{bmatrix}.$$

Finally, the first statement of (vii) follows from Lemma 8(vi), the second statement from Lemma 4(i), and the third statement from the first and $AA^+B = B$. ||

Lemma 10. In the special case when $\mathrm{r}(A:B) = \mathrm{r}(A) + \mathrm{r}(B)$, we have

$$Z^+ = \begin{bmatrix} N^+ - N^+BB'N^+ & N^+B \\ B'N^+ & 0 \end{bmatrix},$$

where $N = A + BB'$, and

$$ZZ^+ = Z^+Z = \begin{bmatrix} NN^+ & 0 \\ 0 & B^+B \end{bmatrix}.$$

Further,

(i) $B'N^+B = B^+B$, $B'N^+A = 0$, $AN^+A = A$
(ii) $N^+ - N^+BB'N^+$ is symmetric and positive semidefinite with rank $\mathrm{r}(A)$
(iii) $\mathrm{r}(Z) = \mathrm{r}(A) + 2\mathrm{r}(B)$
(iv) $|Z| \neq 0$ if and only if $\mathrm{r}(A) = n - k$ and $\mathrm{r}(B) = k$
(v) if Z is non-singular its determinant is $|Z| = (-1)^k|N|$
(vi) $R^+ = N^+ - N^+BB'N^+$, $RR^+ = NN^+ - BB^+$, $AR^+A = A$, where R denotes the $n \times n$ matrix $(I - BB^+)A(I - BB^+)$.

Proof. Since $\mathrm{r}(A:B) = \mathrm{r}(A) + \mathrm{r}(B)$, the statements in (i) follow from Proposition 1. In particular $C = B^+B = C^+$, which leads to the expression for Z^+, using Lemma 3.

Further, (ii) follows from Lemma 8(iii), (iii) from Lemma 4(ii), and (iv) follows from (iii) and Lemma 4(iii). If $|Z| \neq 0$, then $C = I_k$ and hence $|Z| = (-1)^k|N|$ from Lemma 4(iv). Finally, the first statement of (vi) follows from Lemma 8(vi), the second statement from Lemma 4(i), and the third statement from the first and $AN^+A = A$, $AN^+B = 0$. ||

Lemma 11. In the special case when $MAB = 0$, where $M = I - BB^+$, we have

$$Z^+ = \begin{bmatrix} MA^+M & B^{+\prime} \\ B^+ & -B^+AB^{+\prime} \end{bmatrix}$$

and

$$ZZ^+ = Z^+Z = \begin{bmatrix} AMA^+ + BB^+ & 0 \\ 0 & B^+B \end{bmatrix}.$$

Further, letting $\rho = \mathrm{r}(A:B) - \mathrm{r}(B)$,

(i) $MA = MAM = AM$ is symmetric, positive semidefinite with rank ρ
(ii) $MA^+ = MA^+M = A^+M$ is symmetric, positive semidefinite with rank ρ
(iii) $(MA)^+ = MA^+$
(iv) $MAA^+ = MAA^+M = AA^+M$ is symmetric idempotent with rank ρ
(v) $MA^+B = 0$, $MAA^+B = 0$
(vi) $\mathrm{r}(Z) = \rho + 2\mathrm{r}(B)$
(vii) $|Z| \neq 0$ if and only if B has full column-rank k and $\rho = n - k$.

Proof. We first prove (i)–(v), leaving the statements about the ranks for later. (i) follows from $MAB = 0$. Using the symmetry of A and MA, we have

$$\begin{aligned} A^+M &= A^+AA^+M = A^+A^+AM = A^+A^+MA \\ &= A^+A^+MAA^+A = A^+A^+AMA^+A = A^+MAA^+. \end{aligned}$$

Since A^+MAA^+ is symmetric, so is A^+M. Hence $MA^+ = A^+M$. Premultiplying by M completes the proof of (ii). (iii) and (iv) are easily proved using $A = A'$, $MA = AM$, and $MA^+ = A^+M$. The two statements in (v) follow from (ii) and (iv) respectively.

Next, let $R = MAM$. By Lemmas 3 and 4,

$$Z^+ = \begin{bmatrix} R^+ & (I - R^+A)B^{+\prime} \\ B^+(I - AR^+) & -B^+(A - AR^+A)B^{+\prime} \end{bmatrix}$$

and

$$ZZ^+ = Z^+Z = \begin{bmatrix} BB^+ + RR^+ & 0 \\ 0 & B^+B \end{bmatrix}.$$

Then, since

$$R^+ = (MAM)^+ = (MA)^+ = MA^+ = MA^+M,$$

$$R^+AB^{+\prime} = MA^+AB^{+\prime} = A^+AMB(B'B)^+ = 0,$$

and

$$RR^+ = MAMMA^+M = AMA^+,$$

the expressions for Z^+ and ZZ^+ simplify to the ones in the lemma.

It follows from Lemma 8(ii) that $\mathrm{r}(MA^+M) = \mathrm{r}(A:B) - \mathrm{r}(B) = \rho$, and hence that every matrix in (i)–(iv) has rank ρ. Finally, (vi) and (vii) follow from Lemma 4(ii) and 4(iii) respectively. ||

5. The Case $k = 1$

If we think of the matrix Z as arising from constrained optimization, then the case $k = 1$ corresponds to the frequently encountered situation where the optimization is subjected to one constraint only. We shall see that important simplifications occur due to the fact that one of the special cases of Section 4 must apply when $k = 1$. Thus we consider in this section the $(n + 1) \times (n + 1)$ matrix

$$Z = \begin{bmatrix} A & b \\ b' & 0 \end{bmatrix}, \tag{5.1}$$

where A is a real positive semidefinite $n \times n$ matrix and $b \neq 0$ is a real $n \times 1$ vector.

Lemma 12. (Corollary of Lemma 9). Let Z be as in (5.1). In the special case when $b \in \mathcal{M}(A)$, we have

$$Z^+ = (1/\gamma)\begin{bmatrix} \gamma A^+ - A^+bb'A^+ & A^+b \\ b'A^+ & -1 \end{bmatrix},$$

where $\gamma = b'A^+b$, and

$$ZZ^+ = Z^+Z = \begin{bmatrix} AA^+ & 0 \\ 0 & 1 \end{bmatrix}.$$

Further,

(i) $\gamma > 0$

(ii) $\gamma A^{+} - A^{+}bb'A^{+}$ is positive semidefinite with rank $\mathrm{r}(A) - 1$
(iii) $\mathrm{r}(Z) = \mathrm{r}(A) + 1$
(iv) $|Z| \neq 0$ if and only if A is positive definite
(v) $|Z| = -\gamma|A|$.

Proof. Since $b \neq 0$ by assumption, we have $b^{+}b = 1$ and also, using Lemma 9(ii), $\gamma > 0$. The other statements are direct consequences of Lemma 9. ||

Lemma 13. (Corollary of Lemma 10). Let Z be as in (5.1). In the special case when $b \notin \mathscr{M}(A)$, we have

$$Z^{+} = \begin{bmatrix} N^{+} - N^{+}bb'N^{+} & N^{+}b \\ b'N^{+} & 0 \end{bmatrix},$$

where $N = A + bb'$, and

$$ZZ^{+} = Z^{+}Z = \begin{bmatrix} NN^{+} & 0 \\ 0 & 1 \end{bmatrix}.$$

Further,

(i) $b'N^{+}b = 1$, $AN^{+}b = 0$, $AN^{+}A = A$
(ii) $N^{+} - N^{+}bb'N^{+}$ is positive semidefinite with rank $\mathrm{r}(A)$
(iii) $\mathrm{r}(Z) = \mathrm{r}(A) + 2$
(iv) $|Z| \neq 0$ if and only if $\mathrm{r}(A) = n - 1$
(v) $|Z| = -|N|$.

Proof. Since $b \neq 0$ by assumption, we have $b^{+}b = 1$ and $\mathrm{r}(b) = 1$. All statements up to (iv) then follow from Lemma 10. In case $|Z| \neq 0$, (v) follows from Lemma 10(v); in case $|Z| = 0$ we have $|N| = 0$ and (v) holds also. ||

Lemma 14. (Corollary of Lemma 11). Let Z be as in (5.1). In the special case when b is an eigenvector of A, say $Ab = \lambda b$ for some $\lambda \geqslant 0$, we have

$$Z^{+} = (1/b'b)\begin{bmatrix} (b'b)A^{+} - \lambda^{+}bb' & b \\ b' & -\lambda \end{bmatrix}$$

and

$$ZZ^{+} = Z^{+}Z = \begin{bmatrix} AA^{+} + (1-\delta)bb'/b'b & 0 \\ 0' & 1 \end{bmatrix},$$

where $\delta = 0$ if $\lambda = 0$, $\delta = 1$ if $\lambda > 0$. Further,

(i) $A^+ - \lambda^+ bb'/b'b$ is positive semidefinite with rank $\mathrm{r}(A) - \delta$
(ii) $\mathrm{r}(Z) = \mathrm{r}(A) - \delta + 2$
(iii) $|Z| \neq 0$ if and only if $\mathrm{r}(A:b) = n$
(iv) $|Z| = \begin{cases} -(b'b/\lambda)|A| & \text{if } \lambda > 0 \\ -|A + bb'| & \text{if } \lambda = 0. \end{cases}$

Proof. From $Ab = \lambda b$, $b \neq 0$ we establish easily

$$A^+ b = \lambda^+ b,\ b^+ = b'/b'b,\ b^+ b = 1,$$

and

$$M = I - bb'/b'b.$$

For δ as defined above we have of course $\delta = \lambda\lambda^+$. Also,

$$MA^+M = A^+ - \lambda^+ bb'/b'b,\ AMA^+ = AA^+ - \delta bb'/b'b.$$

Next we notice that, given $Ab = \lambda b$, we have $\lambda > 0 \leftrightarrow AA^+b = b \leftrightarrow b \in \mathcal{M}(A)$. Hence $\mathrm{r}(A:b) = \mathrm{r}(A)$ if $\lambda > 0$, and $\mathrm{r}(A:b) = \mathrm{r}(A) + 1$ if $\lambda = 0$. This, in turn, implies

$$\rho \equiv \mathrm{r}(A:b) - \mathrm{r}(b) = \mathrm{r}(A) - \delta.$$

Using these facts, all statements of Lemma 14 up to (iii) follow from Lemma 11. Finally (iv) follows from Lemmas 12(v) and 13(v). ||

6. The Equations $XA = YB' + G$, $XB = H$

In restricted least-squares problems and, more generally, in the minimization of quadratic forms subject to linear constraints, we frequently encounter the following pair of linear equations in X and Y; see, e.g. Magnus and Neudecker (1988, sec. 11.31 and 11.32):

$$XA = YB' + G, \qquad XB = H, \tag{6.1}$$

where A, B, G and H are given matrices (of appropriate orders) and A is positive semidefinite. We can rewrite the equations (6.1) as

$$\begin{bmatrix} A & B \\ B' & 0 \end{bmatrix} \begin{bmatrix} X' \\ -Y' \end{bmatrix} = \begin{bmatrix} G' \\ H' \end{bmatrix}$$

or, for short, $ZX_* = F$. A solution exists if and only if $ZZ^+F = F$, and, if a solution exists, it takes the general form $X_* = Z^+F + (I - Z^+Z)Q$, where Q is arbitrary (Magnus and Neudecker (1988, theorem 2.13)). Applying Lemma 3 and 4(i), we then arrive at the following result.

Proposition 2. The two matrix equations in X and Y given in (6.1) have a solution if, and only if,

$$\mathcal{M}(G') \subset \mathcal{M}(A:B) \text{ and } \mathcal{M}(H') \subset \mathcal{M}(B')$$

in which case the general solution is

$$X = HE' + GD + P(I - NN^+)$$
$$Y = HF - GE + Q(I - B^+B),$$

where

$$D = N^+ - N^+BC^+B'N^+ = R^+$$
$$E = N^+BC^+ = (I - R^+A)(B^+)'$$
$$F = C^+ - CC^+ = B^+(A - AR^+A)(B^+)'$$

and

$$N = A + BB', \; C = B'N^+B, \; R = (I - BB^+)A(I - BB^+).$$

The matrices P and Q are arbitrary matrices of appropriate order.

The special case where $G = 0$ is of particular importance; see Magnus and Neudecker (1988, sec. 3.14 and 11.32). We have

Proposition 3. The matrix equations in X and Y

$$XA = YB', \quad XB = H, \tag{6.2}$$

where A, B and H are given matrices, and A is positive semidefinite, have a solution if, and only if, $\mathcal{M}(H') \subset \mathcal{M}(B')$, in which case the general solution for X is

$$X = H(B'N^+B)^+B'N^+ + P(I - NN^+), \tag{6.3}$$

where $N = A + BB'$ and P is an arbitrary matrix of appropriate order. Moreover,

(i) if $\mathcal{M}(B) \subset \mathcal{M}(A)$, then (6.3) simplifies to

$$X = H(B'A^+B)^+B'A^+ + P(I - AA^+); \tag{6.4}$$

(ii) if $\mathrm{r}(A:B) = \mathrm{r}(A) + \mathrm{r}(B)$, then (6.3) simplifies to

$$X = HB'N^+ + P(I - NN^+); \tag{6.5}$$

(iii) if $(I - BB^+)AB = 0$, then (6.3) simplifies to

$$X = HB^+ + P(I - A(I - BB^+)A^+ - BB^+). \tag{6.6}$$

Proof. The first part of the corollary up to (6.3) is an immediate consequence of Proposition 2. The three special cases follow using Lemmas 9, 10 and 11 respectively. ||

References

Aitchison, J. and S. D. Silvey (1958) Maximum-likelihood estimation of parameters subject to restraints, *Annals of Mathematical Statistics*, **29**, 813–28.

Alalouf, I. S. and G. P. H. Styan (1979) Characterizations of estimability in the general linear model, *The Annals of Statistics*, **7**, 194–200.

Campbell, S. L. and C. D. Meyer (1979) *Generalized Inverses of Linear Transformations*, London: Pitman.

Chipman, J. S. (1964) On least squares with insufficient observations, *Journal of the American Statistical Association*, **59**, 1078–111.

Diewert, W. E. and A. D. Woodland (1977) Frank Knight's theorem in linear programming revisited, *Econometrica*, **45**, 375–98.

Don, F. J. H. (1985) The use of generalized inverses in restricted maximum likelihood, *Linear Algebra and Its Applications*, **70**, 225–40.

Hall, F. J. (1975) Generalized inverses of a bordered matrix of operators, *SIAM Journal of Applied Mathematics*, **29**, 152–63.

Hall, F. J. (1976) On the independence of blocks of generalized inverses of bordered matrices, *Linear Algebra and Its Applications*, **14**, 53–61.

Hall, F. J. and R. E. Hartwig (1976) Further results on generalized inverses of partitioned matrices, *SIAM Journal of Applied Mathematics*, **30**, 617–24.

Hall, F. J. and C. D. Meyer (1975) Generalized inverses of the fundamental bordered matrix used in linear estimation, *Sankhyā*, Series A, **37**, 428–38. Corrigendum in *Sankhyā*, Series A, **40**, 399.

Khatri, C. G. (1968) Some results for the singular normal multivariate regression models, *Sankhyā*, Series A, **30**, 267–80.

Magnus, J. R. (1985) Some Properties of the Bordered Hessian Matrix. Discussion paper 85/126, International Centre for Economics and Related Disciplines, London School of Economics.

Magnus, J. R. and H. Neudecker (1988) *Matrix Differential Calculus with Applications in Statistics and Econometrics*, Chichester/New York: John Wiley.

Marsaglia, G. and G. P. H. Styan (1974) Equalities and inequalities for ranks of matrices, *Linear and Multilinear Algebra*, **2**, 269–92.

Mitra, S. K. (1982) Properties of the fundamental bordered matrix used in linear estimation. In G. Kallianpur, P. R. Krishnaiah and J. K. Ghosh (eds), *Statistics and Probability: Essays in Honor of C. R. Rao*, Amsterdam: North-Holland. 505–9.

Moore, E. H. (1920) On the reciprocal of the general algebraic matrix (Abstract), *Bulletin of the American Mathematical Society*, **26**, 394–5.
Moore, E. H. (1935) *General Analysis*, Part I, Memoirs of the American Philosophical Society, Vol. 1, Philadelphia.
Penrose, R. (1955) A generalized inverse for matrices, *Proceedings of the Cambridge Philosophical Society*, **51**, 406–13.
Plackett, R. L. (1960) *Principles of Regression Analysis*, Oxford: Clarendon Press.
Pringle, R. M. and A. A. Rayner (1971) *Generalized Inverse Matrices With Applications to Statistics*, London: Charles Griffin.
Rao, C. R. (1971) Unified theory of linear estimation, *Sankhyā*, Series A, **33**, 371–94.
Rao, C. R. (1972) A note on the IPM method in the unified theory of linear estimation, *Sankhyā*, Series A, **34**, 285–8.
Rao, C. R. (1973) *Linear Statistical Inference and Its Applications*, 2nd edn, New York: John Wiley.
Rao, C. R. and H. Yanai (1985) Generalized inverses of partitioned matrices useful in statistical applications, *Linear Algebra and Its Applications*, **70**, 105–13.
Samuelson, P. A. (1947) *Foundations of Economic Analysis*, Cambridge, MA: Harvard University Press.
Silvey, S. D. (1959) The Lagrangian multiplier test, *Annals of Mathematical Statistics*, **30**, 389–407.